Numerical simulations of variable density, 3D, swirling flows

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Motivations

- Fundamental aspect: How to solve (numerically) 3D, incompressible and nonhomogeneous, Navier-Stokes equations efficiently (for miscible fluids, ex: He and Air)

- Practical aspect: apply the above solution to study (incompressible) swirling flows with density variations:
  - Swirling flows:
    - Tornadoes, wakes of airplanes and rotor devices in general
    - Industrial applications: increase combustion efficiency, swirl stabilized burners ...

Note that swirling flows present also a fundamental aspect: Vortex breakdown (open problem. Could it be related to the well known open problem of the existence of finite time singularity in Navier-Stokes, or Euler, equations?).
Motion and Mass equations

- Equations

\[
\begin{align*}
\partial_t \rho &= -(u \cdot \nabla) \rho = G(\rho, u), \\
\partial_t u &= -(u \cdot \nabla) u - \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \Delta u + \frac{1}{\rho} f = F(\rho, u) - \frac{1}{\rho} \nabla p, \\
\nabla \cdot u &= 0,
\end{align*}
\]

- Boundary and Initial conditions

\[
\begin{align*}
B(u) &= 0 \quad \text{in} \quad \partial \Omega \times [0, T] \quad \text{(Periodic, here)} \\
u(x, t = 0) &= u_0(x) \quad \text{in} \quad \Omega \\
\rho(x, t = 0) &= \rho_0(x) \quad \text{in} \quad \Omega.
\end{align*}
\]

- Difficulties

- 3D Simulations, \( N = O(10^8) \) grid points
  \( \rightarrow \) The resolution should be parallel
- Pressure and viscous terms are \textit{nonlinear}
  \( \rightarrow \) Necessity to develop adequate solvers
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Spatial discretization: Fourier modes

- Fourier decomposition: \( \mathbf{u}(\mathbf{x}) = \sum \hat{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} \)
- Parallelization: volume decomposition, transposition.
- Communication between processors uses data-blocks with a size not exceeding the cache size of each processor → a *superlinear* speedup is obtained as it will be shown.

\[
\begin{array}{c}
\text{1st direction FFT,} \\
\mathbf{u} \rightarrow \bar{\mathbf{u}}
\end{array}
\quad
\begin{array}{c}
\text{Communication: data exchange,} \\
\bar{\mathbf{u}} \rightarrow \tilde{\mathbf{u}}
\end{array}
\quad
\begin{array}{c}
\text{2nd direction FFT,} \\
\tilde{\mathbf{u}} \rightarrow \hat{\mathbf{u}}
\end{array}
\]
Projection operator

- Constant density, recall that:
  \[
  P = (\mathbb{I}_d - \nabla \Delta^{-1} \nabla \cdot )
  \]

- Variable density, a well known method: artificial compressibility
  \[
  \varepsilon \frac{\partial P}{\partial t} + \nabla \cdot \mathbf{u} = 0
  \]
  \rightarrow \text{Approximate projection operator}
  \rightarrow \varepsilon = \text{is a free parameter! (}\varepsilon \ll 1 \rightarrow \text{numerical instabilities)}

- We propose the following method for variable density:
  \[
  P_\rho = (\mathbb{I}_d - \frac{1}{\rho} \nabla \mathbb{L}_\rho^{-1} \nabla \cdot ), \quad \mathbb{L}_\rho = \nabla \cdot (\frac{1}{\rho} \nabla)
  \]

- Exact projection operator:
  \[
  P_\rho^2 = P_\rho \quad \nabla \cdot P_\rho \mathbf{v} = 0
  \]
Pressure solver

- Equation to be solved (at each timestep):
  \[ \mathbb{L}_\rho P = \nabla \cdot \left( \frac{1}{\rho} \nabla P \right) = h(u) \]

- Two methods:
  - Preconditioned Conjugate-Gradient
    \[ C^{-1/2} \nabla \cdot \left( \frac{1}{\rho} \nabla \right) C^{-1/2} \phi = C^{-1/2} g, \quad C = 1/k^2, \quad \phi = C^{1/2} P \]
  - Fixed point method
    \[ \Delta P^{k+1} = \frac{\nabla P^k \cdot \nabla \rho}{\rho} + \rho h(u) \]

→ Convergence: less than 10 iterations

Temporal discretization

- Runge-Kutta (order 2 or 3)

\[
\rho^{n+1} = \rho^n + \delta t \sum_{i=1}^{p} \beta_i k_i \\
\mathbf{u}^{n+1} = \mathbf{u}^n + \delta t \sum_{i=1}^{p} \beta_i \mathbf{K}_i
\]

\[
\mathbf{K}_p^* = F(\mathbf{u}^n + \delta t \sum_{i=1}^{p-1} \alpha_{p,i} \mathbf{K}_i, \rho^n) \\
K_p = \mathbb{P}_\rho \mathbf{K}_p^*
\]

- Pressure calculation (to conserve the scheme order)

\[
P^n = \sum_{i=1}^{p} \beta_i \mathbb{L}_\rho^{-1} \mathbf{K}_i^*
\]
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Numerical precision

- Temporal precision
- Projection error
- Speed-up

![Graph showing temporal precision](image1)

![Graph showing projection error and speed-up](image2)
Validation

- Instability of a variable density plane jet
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Swirling flow model

- A swirling flow is modeled using a velocity field with an axial component, \( V_z = W_0(r) \), and an azimuthal component, \( V_\theta = V_0(r) \), as a base flow in a cylindrical coordinate system \((r, \theta, z)\).

- We add to this basic flow a density profile \( \rho(r) \), characterized by a density ratio \((s = \rho(0)/\rho(\infty))\), that we use as a control parameter.

- The swirl number, \( q \), is also a control parameter.

Example: variable-density Batchelor vortex
"New" Instability

Helical Rayleigh-Taylor instability \( (s > 1) \)

Impulse response of a variable density swirling jet

- Absolute/Convective transition using the density ratio
Impulse response of a variable density swirling jet

- Convective/Absolute transition for a variable-density Batchelor vortex, for different azimuthal wavenumbers $m$. 

![Graph showing impulse response with different wavenumbers](image-url)
Vortex Breakdown has a tendency to prevent the formation of finite-time blow up in Navier-Stokes equations, but not in Euler equations!

Work in progress

- Statistical properties of incompressible and variable density turbulence.
- Increasing $s$. Actually the method works with $s \leq 10$.
- Include mass diffusion: very complicated equations!
Thank you for your attention